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Abstract

In this paper we study the influence of an electric field on a two dimensional waveguide. We show that bound states that occur under a geometrical deformation of the guide turn into resonances when we apply an electric field of small intensity having a nonzero component on the longitudinal direction of the system.

Keywords: Resonance, Operator Theory, Schrödinger Operators, Waveguide. [1]

1 Introduction

The study of resonances occurring in a quantum system subjected to a constant electric field is now a well-known issue among the mathematical physics community. In a recent past a large amount of literature has been devoted to this problem (see e.g. [15, 17] and references therein). Mostly these works are concerned with quantum systems living in the whole space $\mathbb{R}^n$ as e.g. atomic systems [6, 12, 16, 18, 26, 27].

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In the present paper we would like to address this question for an inhomogeneous quantum system consisting in a curved quantum waveguide in $\mathbb{R}^2$. It is known that bound states arise in curved guides [7, 10] and the corresponding eigenfunctions are expected to be localized in space around the deformation. Therefore, based on these results the main question is what happens with these bound states when the electric field of small intensity is switched on?

A first result is given in [11] where the electric field is supposed to be orthogonal to the guide outside a bounded region. But in this situation there is no Stark resonance.

Here we are focusing on a strip $\mathbf{\Omega} \subset \mathbb{R}^2$ of constant width curved within a compact region. The electric field is chosen with a strictly positive component both on the longitudinal direction of the left part and of the right part of the curved strip. Roughly speaking this situation is similar to the one of an atomic system interacting with an external electric field. Due to the field, an eigenstate of the curved waveguide at zero field turns into scattering state which is able to escape at infinity under the dynamics. It is then natural to expect spectral resonances for this system. In this work we would like to study this question in the weak field regime.

The resonances are defined as the complex poles in the second Riemann sheet of the meromorphic continuation of the resolvent associated to the Stark operator. We construct this extension using the distortion theory [4, 19]. Our proof of existence of resonances borrows elements of strategy developed in [6, 16]. It is mainly based on non-trapping estimates of [6]. For the applicability of these techniques to our model, the difficulty we have to solve is that the system has a bounded transverse direction.

To end this section let us mention a still open question related to this problem and that we hope to solve in a future work. We claim that our regularity assumptions on the curvature imply that the corresponding Stark operator (see (2.4)) has no real eigenvalue [5]. In that case the complex poles have a non zero imaginary part then they are resonances in the strict sense of the term [24].

Let us briefly review the content of the paper. In section 2 we describe precisely the system, assumptions and the main results. The distortion and the definition of resonances are given respectively in section 3 and 4. In section 5 we prove the existence of resonances. Finally the section 6 is devoted to get an exponential estimate on the width of resonances. Actually we show that the imaginary part of resonances arising in this system follows a type of Oppenheimer's law [22] when the intensity of the field vanishes.

2 Main results

2.1 Setting

Before describing the main results of the paper we want to recast the problem into a more convenient form. This allows us to state precisely our assumptions on the system.

Consider a curved strip $\mathbf{\Omega}$ in $\mathbb{R}^2$ of a constant width d defined around a smooth reference curve Γ, we suppose that $\mathbf{\Omega}$ is not self-intersecting. The points $\mathbf{X} = (x, y)$ of $\mathbf{\Omega}$ are described by the curvilinear coordinates $(s, u) \in \mathbb{R} \times (0, d)$,

$$\begin{aligned} x &= a(s) - ub'(s), \\ y &= b(s) + ua'(s), \end{aligned} \tag{2.1}$$

where a, b are smooth functions defining the reference curve $\Gamma = \{(a(s), b(s)), s \in \mathbb{R}\}$ in $\mathbb{R}^2$. They are supposed to satisfy $a'(s)^2 + b'(s)^2 = 1$.

Introduce the signed curvature $\gamma(s)$ of Γ,

$$\gamma(s) = b'(s)a''(s) - a'(s)b''(s). \tag{2.2}$$

For a given curvature γ, the functions a and b can be chosen as

$$a(s) = \int_0^s \cos\alpha(t)\, dt,\ b(s) = \int_0^s \sin\alpha(t)\, dt, \tag{2.3}$$

where $\alpha(s_1, s_2) = -\int_{s_2}^{s_1} \gamma(t)\, dt$ is the angle between the tangent vectors to Γ at the points s_1 and s_2 (See e.g. [10] or [11] for more details). Set $\alpha(s) = \alpha(s, 0), s \in \mathbb{R}$ and $\alpha_0 = \alpha(s_0)$. We choose γ with a compact support, $\mathrm{supp}(\gamma) = [0, s_0]$ for some $s_0 > 0$. In particular for $s < 0$ the strip is straight, parallel to the $x-$axis. Assume also that

(h1) $\gamma \in \mathrm{C}^2(\mathbb{R})$,

(h2) $d\|\gamma\|_\infty < 1$.

Evidently this implies that γ has a continuous and bounded derivative up to second order.

Let $\mathbf{F} = F(\cos(\eta), \sin(\eta))$ be the electric field. In this work the intensity of the field $F > 0$ is the free parameter and the direction η is fixed. It satisfies

(h3) $|\eta| < \frac{\pi}{2}$ and $|\eta - \alpha_0| < \frac{\pi}{2}$.

See the Remark 2.1 below for a discussion about assumptions on η. We consider the Stark effect Hamiltonian on $L^2(\mathbf{\Omega})$,

$$\mathbf{H}(F) = -\Delta_{\mathbf{\Omega}} + \mathbf{F} \cdot \mathbf{X},\ F > 0, \tag{2.4}$$

with Dirichlet boundary conditions on $\partial\mathbf{\Omega}$, the boundary of $\mathbf{\Omega}$. One can check that under (h1) and (h2), then by using natural curvilinear coordinates, $\mathbf{H}(F)$ is unitarily equivalent to the Schrödinger operator defined by

$$H(F) = H_0(F) + V_0, \quad H_0(F) = H_0 + W(F), \quad H_0 = T_s + T_u \tag{2.5}$$

on the Hilbert space $L^2(\Omega)$, $\Omega = \mathbb{R} \times (0, d)$ with Dirichlet boundary conditions on $\partial\Omega = \mathbb{R} \times \{0, d\}$. Here

$$T_s = -\partial_s g \partial_s, \; g = g(s, u) = (1 + u\gamma(s))^{-2}, \; T_u = -\partial_u^2 \tag{2.6}$$

and $W(F)$ is the operator multiplication by the function,

$$W(F, s, u) = \begin{cases} F(\cos(\eta)s + \sin(\eta)u) & \text{if } s < 0 \\ F(\int_0^s \cos(\eta - \alpha(t))\, dt + \sin(\eta - \alpha(s))u) & \text{if } 0 \le s \le s_0 \\ F(\cos(\eta - \alpha_0)(s - s_0) + A + \sin(\eta - \alpha_0)u) & \text{if } s > s_0 \end{cases} \tag{2.7}$$

where $A = \int_0^{s_0} \cos(\eta - \alpha(t))\, dt$
and

$$V_0(s, u) = -\frac{\gamma(s)^2}{4(1 + u\gamma(s))^2} + \frac{u\gamma''(s)}{2(1 + u\gamma(s))^3} - \frac{5}{4}\frac{u^2\gamma'(s)^2}{(1 + u\gamma(s))^4}. \tag{2.8}$$

Denote by $H = H_0 + V_0$. This is the hamiltonian associated with the guide in absence of electric field. If (h1) and (h2) are satisfied, then H is a self-adjoint operator on $L^2(\Omega)$ with domain $D(H)$ coinciding with the one of H_0 and [20]

$$D(H) = D(H_0) = \{\varphi \in \mathcal{H}_0^1(\Omega), \; H_0\varphi \in L^2(\Omega)\} \tag{2.9}$$

In this paper we use standard notation from Sobolev space theory. The essential spectrum of H, $\sigma_{ess}(H) = [\lambda_0, +\infty)$ where λ_0 is the first transverse mode of the system i.e the first eigenvalue of the operator T_u on $L^2(0, d)$ with Dirichlet boundary conditions at $\{0, d\}$.

Moreover under our assumptions the operator H has at least one discrete eigenvalue below the essential spectrum (see [10]). Although we do not know the discrete spectrum of H our study below works even in the case where H has infinitely many distinct discrete eigenvalues (possibly degenerate) which can accumulate at the threshold λ_0.

Remark 2.1. *The situation where $\eta = \frac{\pi}{2}$ and $\alpha_0 = 0$ has been considered in [11], but in that case there is no Stark resonance. It is also true if we suppose $|\eta| \geq \frac{\pi}{2}$ and $|\eta - \alpha_0| < \frac{\pi}{2}$ since $W(F)$ is now a confining potential. Note that the regime $|\eta| > \frac{\pi}{2}$ and $|\eta - \alpha_0| > \frac{\pi}{2}$ is a symmetric case of the one considered in this paper and can be studied in the same way. While for $|\eta| < \frac{\pi}{2}$ and $|\eta - \alpha_0| > \frac{\pi}{2}$, the situation is quite different since $W(F) \to -\infty$ at both $s \to \pm\infty$ the "escape" region corresponding to any negative energy contains $\{s < -a, u \in (0, d)\} \cup \{s > a, u \in (0, d)\}$ for some $a > 0$ and large. This needs slight modifications of our method. It is actually studying in [14].*

2.2 Results

In this section we give the main results of the paper. Some minor points will be specified later in the text.

First we need to define rigorously the Stark Hamiltonian associated to our system. Here we adopt a common fact in the literature about Stark operators i.e. $H(F)$ is well defined as an essentially self-adjoint operator on $L^2(\Omega)$ [8, 20, 23]. In the appendix of the paper where the proof of the theorem below is proved, we give a core for $H(F)$.

Theorem 2.1. *Suppose that (h1) and (h2) hold, then for $F > 0$,*

(i) $H(F)$ is an essentially self-adjoint operator on $L^2(\Omega)$, We will denote the closure of $H(F)$ by the same symbol.

(ii) The spectrum of $H(F)$, $\sigma(H(F)) = \mathbb{R}$.

We now are focusing on the second main result and its proof. For any subset $\mathcal{D}$ of $\mathbb{C}$, denote by $\mathcal{D}^- = \{z \in \mathcal{D}, \mathrm{Im} z \leq 0\}$.

Theorem 2.2. *Suppose that (h1), (h2) and (h3) hold. Let E_0 be an discrete eigenvalue of H of finite multiplicity $j \in \mathbb{N}$. There exits $F_c > 0$, a F-independent complex neighbourhood ν_{E_0} of the semi axis $(-\infty, E_0 + \frac{1}{8}(\lambda_0 - E_0)]$ and a F-independent dense subset $\mathcal{A}$ of $L^2(\Omega)$such that for $0 < F \leq F_c$*

i) the function

$$z \in \mathbb{C}, \mathrm{Im} z > 0 \to \mathcal{R}_\varphi(z) = \big((H(F) - z)^{-1}\varphi, \varphi\big),\ \varphi \in \mathcal{A}$$

has an meromorphic extension in ν_{E_0} through the cut due to the spectrum of $H(F)$.
ii) $\cup_{\varphi \in \mathcal{A}}\{\text{poles of } \mathcal{R}_\varphi(z)\} \cap \nu_{E_0}^-$ contains j poles $Z_0(F), ...Z_j(F)$ converging to E_0 when $F \to 0$.

Here resonances of the stark operator $H(F)$ are defined as the set [24]

$$\cup_{\varphi \in \mathcal{A}}\{\text{poles of } \mathcal{R}_\varphi(z)\} \cap \mathbb{C}^-.$$

The resonances have necessarily a negative imaginary part. But as it is discussed in the introduction, a still open question is concerned with the strict negativity.

We shall show below that using the distortion theory, the resonances coincide with discrete eigenvalues of a non self-adjoint operator.

Finally we get the following exponential bound on the width of resonances.

Theorem 2.3. *Under conditions of the Theorem 2.2. Let E_0 be a simple eigenvalue of H and Z_0 the corresponding resonance for $H(F)$ given by the Theorem 2.2. Then there exist two constants $0 < c_1, c_2$ such that for $0 < F \le F_c$,*

$$|\mathrm{Im} Z_0| \le c_1 e^{-\frac{c_2}{F}},\ k = 1, .., j.$$

Remark 2.2. *Our results exhibit a critical field value F_c. For the Theorem 2.2 i) this value is estimated explicitly (see formula* (4.25) *below). But this is not true for the rest of the results since our method use certain abstract analysis arguments which are valid for F small enough. This does not give an explicit critical F_c.*

3 The distortion theory

In this section, by using the distortion theory we construct a family of non self adjoint operators $\{H_\theta(F), \theta \in \mathbb{C}, |\mathrm{Im}\theta| < \theta_0\}$ for some $\theta_0 > 0$. In the next section we will see that under conditions the discrete spectrum of $H_\theta(F)$ coincides with resonances of $H(F)$. We refer the reader to [4, 19, 23] for basic tools of the distortion theory. Here we assume that (h1), (h2) and (h3) are satisfied. To give a sense to the construction below we need to consider electric fields of finite magnitude. Without loss of generality we may suppose in the sequel that $0 < F \le 1$.

Introduce the distortion on Ω,

$$S_\theta : (s, u) \mapsto (s + \theta f(s), u) \tag{3.10}$$

defined from the vector field $f = -\frac{1}{F\cos(\eta)}\Phi$ where $\Phi \in \mathrm{C}^\infty(\mathbb{R})$ is as follow. Let $E < 0$, be the reference energy, $0 < \delta E < \frac{1}{2}\min\{1, |E|\}$, $E_- = E - \delta E$ and $E_+ = E + \delta E$. Set $\Phi(s) = \phi[F\cos(\eta)s]$ where $\phi \in \mathrm{C}^\infty(\mathbb{R})$ is a non-increasing function such that

$$\phi(t) = 1 \text{ if } t < E,\ \phi(t) = 0 \text{ if } t > E_+ \tag{3.11}$$

and satisfying $\|\phi^{(k)}\|_\infty = o((\frac{1}{\delta E})^k$. Note that for $s < \frac{E}{F\cos(\eta)}$, S_θ coincides with a translation w.r.t. the longitudinal variable s.

Clearly for $k \ge 1$, $\|\Phi^{(k)}\|_\infty \le (\frac{F}{\delta E})^k$ and $\|f^{(k)}\|_\infty \le \frac{F^{k-1}}{(\delta E)^k}$. For $\theta \in \mathbb{R}$, $|\theta| < \delta E$, S_θ implements a family of unitary operators on $L^2(\Omega)$ by

$$U_\theta \psi = (1 + \theta f')^{\frac{1}{2}} \psi \circ S_\theta. \tag{3.12}$$

We note that

$$H_\theta(F) = U_\theta H(F) U_\theta^{-1} = H_{0,\theta}(F) + V_0. \tag{3.13}$$

$$H_{0,\theta}(F) = T_{s,\theta} + T_u + W_\theta(F) \tag{3.14}$$

where

$$T_{s,\theta} = -(1+\theta f')^{-\frac{1}{2}} \partial_s (1+\theta f')^{-1} g \partial_s (1+\theta f')^{-\frac{1}{2}}, \tag{3.15}$$

$$W_\theta(F) = W(F) \circ S_\theta. \tag{3.16}$$

We now want to extend the definition of $H_\theta(F)$ for complex θ. Set $\theta_0 = \alpha\delta E$ where α is a some small and strictly positive constant which we fix in the proof of the Theorem 3.1 below. In fact θ_0 is the critical value of distortion parameter.

Proposition 3.1. *There exists $0 < \alpha < 1/2$ independent of E and F such that for $0 < F < \delta E$, $\{H_\theta(F), |\mathrm{Im}\theta| < \theta_0\}$ is a self-adjoint analytic family of operators (see [20]).*

Proof. An computation shows that

$$T_{s,\theta} = -\partial_s (1+\theta f')^{-2} g \partial_s + R_\theta, \tag{3.17}$$

where $R_\theta = \frac{g}{2}\frac{\theta f'''}{(1+\theta f')^3} - \frac{5g}{4}\frac{\theta^2 f''^2}{(1+\theta f')^4}$ is a bounded function. Let $h(F) = H_0 + w(F)$ be the operator in $L^2(\Omega)$ where $w(F)$ is the multiplication operator by

$$w(F,s) = \begin{cases} F\cos(\eta)s & \text{if } s < 0 \\ 0 & \text{if } 0 \le s \le s_0 \\ F\cos(\eta - \alpha_0)s & \text{if } s > s_0. \end{cases} \tag{3.18}$$

Since $h = h(F)$ differs from $H(F)$ by adding a bounded symmetric operator, it is also a self-adjoint operator.

We have

$$H_\theta(F) = h + \partial_s G_\theta \partial_s + R_\theta + W_\theta(F) - w(F) + V_0,\ G_\theta = g\big(\frac{2\theta f' + \theta^2 f'^2}{(1+\theta f')^2}\big). \tag{3.19}$$

Let us show that for $|\theta|$ small enough then $D(H_{0,\theta}(F)) = D(h)$. Through unitary equivalence we may suppose that $\mathrm{Re}\theta = 0$. In view of the perturbation theory [20] and (3.19)we only need to to show that $\partial_s G_\theta \partial_s$ is h-bounded with a relative bound strictly smaller than one. By using the resolvent identity,

$$\begin{aligned} \partial_s G_\theta \partial_s (h+i)^{-1} &= \partial_s G_\theta \partial_s (H_0+i)^{-1} - \partial_s G_\theta \partial_s (H_0+i)^{-1} w(F)(h+i)^{-1} \\ &= \partial_s G_\theta \partial_s (H_0+i)^{-1} - \partial_s G_\theta \partial_s F s (H_0+i)^{-1} \frac{w(F)}{Fs}(h+i)^{-1} \\ &- \partial_s G_\theta \partial_s (H_0+i)^{-1} (\partial_s g + g\partial_s)(H_0+i)^{-1} \frac{w(F)}{s}(h+i)^{-1}. \end{aligned}$$

We know that $D(H_0) \subset \mathcal{H}^2_{\rm loc}(\bar{\Omega}) \cap \mathcal{H}^1_0(\Omega)$, [2, 9, 21]. Let χ be a characteristic function of $\mathrm{supp}(f')$. Then by the closed graph theorem [20] $\partial_s g(H_0+i)^{-1}$, $g\partial_s(H_0+i)^{-1}$, $\chi\partial_s g\partial_s(H_0+i)^{-1}$ and $\chi\partial_s g\partial_s Fs(H_0+i)^{-1}$ are bounded operators. The multiplication operators $\frac{w(F)}{s}$ and $\frac{w(F)}{Fs}$ are also bounded. Hence this is true for the operator $\partial_s G_\theta \partial_s(h+i)^{-1}$.

It is easy to check that under conditions on parameters θ and F,

$$\|\partial_s G_\theta \partial_s(H_0+i)^{-1}\| \leq \frac{3\alpha}{(1-\alpha)^3}\big(\|\chi\partial_s g\partial_s(H_0+i)^{-1}\|+\|g\partial_s(H_0+i)^{-1}\|\big) \leq C\frac{3\alpha}{(1-\alpha)^3}$$

for some constant $C>0$ independent of F and E. Evidently $\|\partial_s G_\theta\partial_s Fs(H_0+i)^{-1}\|$ satisfies a similar estimate. Choosing α so small such that $\|\partial_s G_\theta \partial_s(h+i)^{-1}\|<1$, then $\partial_s G_\theta \partial_s$ is relatively bounded to h with relative bound strictly smaller that one. Thus the statement follows.

The proof is complete if we can show that for $\psi \in D(H_\theta(F))$

$$\theta \in \{\theta \in \mathbb{C}, |\theta| < \theta_0\} \longmapsto (H_\theta(F)\psi, \psi)$$

is an analytic function. But this last fact can be readily verified by using standard arguments of measure theory and the explicit expression (3.19) (see e.g. [23, 20]).

$\square$

Remark 3.3. *For $\theta \in \mathbb{R}$, $|\theta| < \delta E$ consider the unitary transformation on $L^2(\mathbb{R})$*

$$u_\theta\psi(s) = (1+\theta f'(s))^{\frac{1}{2}}\psi(s+\theta f(s)),\ \psi \in L^2(\mathbb{R}).$$

We know from [19, 24] that there exists a dense subset of analytic vectors ψ associated with u_θ in $|\theta| < \frac{\delta E}{\sqrt{2}}$ i.e. $\theta \in \mathbb{R} \to u_\theta\psi$ has an $L^2(\mathbb{R})$-analytic extension in $|\theta| < \frac{\delta E}{\sqrt{2}}$. denote this set as $\mathcal{A}_1$. It is shown in [19] that $\mathcal{A}_1$ is dense in $L^2(\mathbb{R})$. Let $\mathcal{A}$ be the linear subspace generated by vectors of the form $\varphi\otimes\psi, \varphi \in \mathcal{A}_1, \psi \in L^2\big((0,d)\big)$. Then $\mathcal{A}$ is a dense subset of analytic vectors associated to the transformation U_θ in $|\theta| < \theta_0$.

For further developments we need to introduce the following modified operator on $L^2(\Omega)$. Let $s_1 > s_0$, such that $\cos(\eta-\alpha_0)(s-s_0)+A+\sin(\eta-\alpha_0)u \geq 0$ for all $u \in (0,d)$. Set

$$\widetilde{H}_0(F) = H_0 + \widetilde{W}(F), \tag{3.20}$$

where $\widetilde{W}(F)$ is a multiplication operator by

$$\widetilde{W}(F,s,u) = \begin{cases} W(F,s,u) & \text{if } s<0,\ s>s_1 \\ \\ 0 & \text{if } 0 \leq s \leq s_1. \end{cases} \tag{3.21}$$

For $\theta \in \mathbb{R}$, $|\theta| < \theta_0$, let $\widetilde{H}_{0,\theta}(F) = U_\theta \widetilde{H}_0(F) U_\theta^{-1} = T_{s,\theta} + T_u + \widetilde{W}_\theta(F)$. Then we have

Corollary 3.1. *For $0 < F < \delta E$, $\{\tilde{H}_{0,\theta}(F),\ |\mathrm{Im}\theta| < \theta_0\}$ is a self-adjoint analytic family of operators.*

Proof. We have $H_\theta(F) - \widetilde{H}_{0,\theta}(F) = V_0 + W_\theta(F) - \widetilde{W}_\theta(F)$, but V_0 as well as $W_\theta(F) - \widetilde{W}_\theta(F)$ are bounded and θ-independent so by the Proposition 3.1 the corollary follows. □

4 Meromorphic extension of the resolvent.

Let $\theta = i\beta$, we suppose that $0 < \beta < \theta_0$. Set

$$\mu_\theta = 1 + \theta f^\sharp \tag{4.22}$$

with $f^\sharp = \Phi - 1$ and Φ defined in the Section 3. Then the multiplier operator by the function μ_θ defined a one to one map from $D(H_\theta(F))$ to $D(H_\theta(F))$. Recall that $\lambda_0 = inf\sigma(T_u)$ is the first transverse mode. Let

$$\nu_\theta = \{z \in \mathrm{C},\ \mathrm{Im}\mu_\theta^2(E_- + \lambda_0 - z) < \beta\frac{\delta E}{2}\}. \tag{4.23}$$

ν_θ^c denotes its complement in $\mathbb{C}$. It is easy to see that ν_θ contains a F-independent complex neighbourhood of the semi axis $(-\infty, \lambda_0 + E - \frac{3}{4}\delta E]$ denoted by $\tilde{\nu}_\theta$. It is defined as

$$\tilde{\nu}_\theta = \{x \le 0, y \ge -\frac{\beta\delta E}{2}\} \cup \{(x > 0, y \ge 2\beta x - \frac{\beta\delta E}{2}\} \tag{4.24}$$

where $x = \mathrm{Re}z - \lambda_0 - E_-$ and $y = \mathrm{Im}z$.

In this section our main result is the following. Let

$$F_0 = \alpha'(\delta E)^2 \min\{1, 1/d\} \tag{4.25}$$

where α' is a strictly positive constant independent of E and β which is determined in the proof of the Lemma 4.1. We have

Proposition 4.2. *There exits $\alpha' > 0$ such that for all $E < 0$, $0 < F \le F_0$, the function*

$$z \in \mathbb{C}, \mathrm{Im}z > 0 \to \mathcal{R}_\varphi(z) = \big((H(F) - z)^{-1}\varphi, \varphi\big),\ \varphi \in \mathcal{A}$$

has an meromorphic extension in $\cup_{0<\beta<\theta_0}\nu_\beta$.

As a consequence of the Proposition 4.2, the Theorem 2.2 i) is proved.

The proof of the Proposition 4.2 is based on the two following results. For a given operator O on $L^2(\Omega)$ we denote by $\varrho(O)$ its the resolvent set.

Lemma 4.1. *There exits $\alpha' > 0$ such that for $E < 0$, $0 < F \le F_0$ and $0 < \beta < \theta_0$. Then* *(i)* $\nu_\theta \subset \varrho(\widetilde{H}_{0,\theta}(F))$.

(ii) $\forall z \in \nu_\theta$, $\|(\widetilde{H}_{0,\theta}(F) - z)^{-1}\| \le \mathrm{dist}^{-1}(z, \nu_\theta^c)$.

Proof. By using a standard commutation relation we derive from (3.15),

$$\mu_\theta T_{s,\theta} \mu_\theta = T_1(\theta) + iT_2(\theta) + \mu_\theta(T_{s,\theta}\mu_\theta) \tag{4.26}$$

where $T_1(\theta) = -\partial_s \mathrm{Re}\{\mu_\theta^2(1+\theta f')^{-2}\} g\partial_s$, $T_2(\theta) = -\partial_s \mathrm{Im}\{\mu_\theta^2(1+\theta f')^{-2}\} g\partial_s$. The operators $T_1(\theta)$, $T_2(\theta)$ are symmetric and we know from [6] that $T_2(\theta)$ is negative. Moreover a straightforward calculation shows

$$\mathrm{Im}\mu_\theta(T_{s,\theta}\mu_\theta) = O\left(\frac{\beta F^2}{(\delta E)^3}\right). \tag{4.27}$$

In the other hand, let $z \in \nu_\theta$, set $\beta S = -\mathrm{Im}\mu_\theta^2(\widetilde{W}_\theta(F) - E_-) - \mathrm{Im}\mu_\theta(T_{s,\theta}\mu_\theta)$ in fact

$$S = (1 - \beta^2 f^{\sharp 2})\Phi - 2f^\sharp\big(\widetilde{W}(F) - E_-\big) - \beta^{-1}\mathrm{Im}\mu_\theta(T_{s,\theta}\mu_\theta). \tag{4.28}$$

On $\mathrm{supp}(f^\sharp) = \mathrm{supp}(\Phi - 1)$, we have $\cos(\eta - \alpha_0)(s - s_0) + \sin(\eta - \alpha_0)u + A \ge 0$ if $s > s_1$, $F\cos(\eta)s - E_- \ge \delta E$ if $s < 0$ and then

$$F\cos(\eta)s\chi_{\{s<0\}} + F(\cos(\eta-\alpha_0) + \sin(\eta-\alpha_0)u + A)\chi_{\{s\ge s_1\}} - E_- \ge \delta E\chi_{\{s<0\}} - E_-\chi_{\{s\ge 0\}} \ge \delta E.$$

By using (4.27), we get for $0 < \beta < \theta_0$

$$S \ge \frac{1}{2}\Phi + 2(1-\Phi)(\delta E + Fu\sin\eta\chi_{\{s<0\}}) + O\left(\frac{F^2}{(\delta E)^3}\right).$$

Then we can choose α' so small such that,

$$S \ge \frac{1}{2}\min\{\frac{1}{2}, \delta E\} = \frac{\delta E}{2}.$$

Further in the quadratic form sense on $D(H_\theta(F)) \times D(H_\theta(F))$, we have

$$Im\mu_\theta(\widetilde{H}_{0,\theta}(F) - z)\mu_\theta = T_2(\theta) - \beta S + \mathrm{Im}\mu_\theta^2 T_u + Im\mu_\theta^2(E_- - z). \tag{4.29}$$

Thus for $0 < \beta < \theta_0$, $0 < F \le F_0$ and $z \in \nu_\theta$, since $\mathrm{Im}\mu_\theta^2 = 2\beta f^\sharp \le 0$, we get

$$Im\mu_\theta(\widetilde{H}_{0,\theta}(F) - z)\mu_\theta \;\; \le \;\; -\beta\frac{\delta E}{2} + \mathrm{Im}\mu_\theta^2(E_- + \lambda_0 - z) < 0. \tag{4.30}$$

This last estimate with together some usual arguments for non-trapping estimates given in [6] complete the proof of the Lemma (4.1). □

Introduce the following operator, let $\theta \in \mathbb{C}$, $|\theta| < \theta_0$ and $z \in \nu_\theta$

$$K_\theta(F,z) = (V_0 + W_\theta(F) - \widetilde{W}_\theta(F))(\widetilde{H}_{0,\theta}(F) - z)^{-1}. \tag{4.31}$$

Lemma 4.2. *In the same conditions as in the previous lemma.*

(i) $z \in \nu_\theta \to K_\theta(F,z)$ *is an analytic compact operator valued function.*

(ii) *For* $z \in \nu_\theta$, $Imz > 0$ *large enough,* $\|K_\theta(F,z)\| < 1$.

Proof. By the Lemma 4.1, (i) follows if we show that $K_\theta(F,z), z \in \nu_\theta$ are compact operators. Set $V = V_0 + W_\theta(F) - \widetilde{W}_\theta(F)$. Notice that V has compact support in the longitudinal direction and it is a bounded operator.

Introduce the operator $\tilde{h} = \tilde{h}(F) = H_0 + \tilde{w}(F)$ on $L^2(\Omega)$ where $\tilde{w}(F)$ is the multiplication operator by

$$\tilde{w}(F,s) = \begin{cases} F\cos(\eta)s & \text{if } s < 0 \\ 0 & \text{if } 0 \le s \le s_1 \\ F\cos(\eta - \alpha_0)s & \text{if } s > s_1. \end{cases} \tag{4.32}$$

Then

$$\widetilde{H}_{0,\theta}(F) - \tilde{h} = \partial_s G_\theta \partial_s + R_\theta + \tilde{W}_\theta(F) - \tilde{w}(F), \tag{4.33}$$

where R_θ, G_θ and $\widetilde{W}_\theta(F)$ are defined in the Section 3. Suppose $|\theta| < \theta_0$, $0 < F < \delta E$, this is satisfied under assumptions of the lemma. Then following step by step the proof of the Proposition3.1, $\widetilde{H}_{0,\theta}(F) - \tilde{h}$ is $\tilde{h}$-bounded with a relative bound smaller than one. Therefore, to prove (i) we are left to show that for $z \in \nu_\theta, \mathrm{Im}z \neq 0$, $V(\tilde{h} - z)^{-1}$ is compact.

Denote by $\mathbb{I}_{\mathcal{H}}$ the identity operator on the space $\mathcal{H}$. Let $h_0 = -\partial_s^2 \otimes \mathbb{I}_{L^2(0,d)} + \mathbb{I}_{L^2(\mathbb{R})} \otimes T_u$ and $G = g - 1$, we have

$$V(\tilde{h} - z)^{-1} = V(h_0 - z)^{-1} + V(h_0 - z)^{-1}\big(\partial_s G \partial_s - \tilde{w}(F)\big)(\tilde{h} - z)^{-1} \tag{4.34}$$

Note that by using again the Herbst's argument [16], the second term of the r.h.s of (4.34) can be written as

$$V(h_0 - z)^{-1}\tilde{w}(F)(\tilde{h} - z)^{-1} = Vs(h_0 - z)^{-1}\frac{\tilde{w}(F)}{s}(\tilde{h} - z)^{-1} + \\ V(h_0 - z)^{-1}[s, h_0](h_0 - z)^{-1}\frac{\tilde{w}(F)}{s}(\tilde{h} - z)^{-1}. \tag{4.35}$$

In the one hand let χ be a C^∞ characteristic function of $[0, s_1]$ then $\chi(h_0 - z)^{-1}$ is a compact operator. Indeed,

$$\chi(h_0 - z)^{-1} = \sum_{n\geq 0} \chi(-\partial_s^2 + \lambda_n - z)^{-1} \otimes p_n$$

where $\lambda_n, n \in \mathbb{N}$ are the eigenvalues of the operator T_u (transverse modes) and $p_n, n \in \mathbb{N}$ the associated projectors. We know that $\chi(-\partial_s^2 + \lambda_n - z)^{-1} \otimes p_n$ is compact [23] and for large n,

$$\|\chi(-\partial_s^2 + \lambda_n - z)^{-1} \otimes p_n\| \leq \|(-\partial_s^2 + \lambda_n - z)^{-1}\| = O(\frac{1}{n^2}). \tag{4.36}$$

Thus $\chi(h_0 - z)^{-1}$ is compact since it is a limit of a sequence of compact operators in the norm topology. This holds true for operators $V(h_0 - z)^{-1}$ and $Vs(h_0 - z)^{-1}$.

On the other hand the function G has a bounded support in the longitudinal direction then the same arguments as in the proof of the Proposition 3.1 imply that the operator $\partial_s G \partial_s(\tilde{h} - z)^{-1}$ is bounded. By the closed graph theorem $[s, h_0](h_0 - z)^{-1} = 2\partial_s(h_0 - z)^{-1}$ is also bounded.

Then by (4.34) and (4.35) the statement follows.

The assertion (ii) is a direct consequence of the Lemma 4.1 (ii) and the fact that V is a bounded operator. □

4.1 Proof of the Proposition 4.2

Here we refer e.g. to [24] for the reader unfamiliar with the distortion theory.

Let $E < 0$, $|\theta| < \theta_0$ and $0 < F \leq F_0$. By Lemmas 4.1, 4.2 and the standard Fredholm alternative theorem, the operator $\mathbb{I}_{L^2(\Omega)} + K_\theta(F, z)$ is invertible for all $z \in \nu_\theta \setminus \mathcal{R}$ where $\mathcal{R}$ is a discrete set. In the bounded operator sense, we have

$$(H_\theta(F) - z)^{-1} = (\tilde{H}_{0,\theta}(F) - z)^{-1}\big(\mathbb{I}_{L^2(\Omega)} + K_\theta(F, z)\big)^{-1}. \tag{4.37}$$

This implies that $\nu_\theta \setminus \mathcal{R} \subset \rho(H_\theta(F))$.

Further let $\mathcal{O}$ an open subset of $\nu_\theta \setminus \mathcal{R}$. For $\varphi \in \mathcal{A}$, consider the function

$$z \in \mathcal{O} \to \mathcal{R}_\varphi(z) = \big((H(F) - z)^{-1}\varphi, \varphi\big). \tag{4.38}$$

For $\theta \in \mathbb{R}$, $|\theta| < \theta_0$, by using the identity $U_\theta^* U_\theta = \mathbb{I}_{L^2(\Omega)}$ in the scalar product of the r.h.s. of (4.38), we have $\mathcal{R}_\varphi(z) = \big((H_\theta(F) - z)^{-1}\varphi_\theta, \varphi_\theta\big)$, $\varphi_\theta = U_\theta\varphi$. Then together with the Proposition 3.1, it holds

$$\mathcal{R}_\varphi(z) = \big((H_\theta(F) - z)^{-1}\varphi_\theta, \varphi_{\bar{\theta}}\big). \tag{4.39}$$

in the disk $\{\theta \in \mathbb{C}, |\theta| < \theta_0\}$.

Fix $\theta = i\beta, 0 < |\beta| < \theta_0$ then $\mathcal{R}_\varphi$ has an meromorphic extension in ν_θ given by

$$\mathcal{R}_\varphi(z) = \big((\tilde{H}_{0,\theta}(F) - z)^{-1}\big(\mathbb{I}_{L^2(\Omega)} + K_\theta(F,z)\big)^{-1}\varphi_\theta, \varphi_{\bar{\theta}}\big).$$

The poles of $\mathcal{R}_\varphi$ are locally θ-independent. From [19] and standard arguments, these poles are the set of $z \in \nu_\theta$ such that the equation $K_\theta(F,z)\psi = -\psi$ has non-zero solution in $L^2(\Omega)$. In view of (4.37) they are the discrete eigenvalues of the operator $H_\theta(F)$. □

5 Resonances.

This section is devoted to the proof ii) of the Theorem 2.2. In view of the section 4.1 It is given by the following

Proposition 5.3. *Let E_0 be an discrete eigenvalue of H of finite multiplicity $j \in \mathbb{N}$. There exists $0 < F_0' \le F_0$ such that for $0 < F \le F_0'$, the operator $H_\theta(F)$, $0 < |\theta| < \theta_0$ has j eigenvalues near E_0 converging to E_0 as $F \to 0$.*

We need first to show the following result. For $\text{Im}z \neq 0$ let $K(z) = V_0(H_0 - z)^{-1}$ They are compact operators (see e.g. arguments developed in the Section 6). Note that formally $K(z) = K_\theta(F = 0, z)$. We have

Lemma 5.3. *Let $E < 0$, $\theta = i\beta$, $0 < \beta < \theta_0$. Let κ be a compact subset of $\tilde{\nu}_\theta \cap \rho(H_0)$, $\chi = \chi(s) \in C_0^\infty(\mathbb{R}^+)$. Then*

(i) $\lim_{F\to 0} \|(\widetilde{H}_{0,\theta}(F) - z)^{-1}\psi - (H_0 - z)^{-1}\psi\| = 0,\ \psi \in L^2(\Omega)$,

(ii) $\lim_{F\to 0} \|\chi(\widetilde{H}_{0,\theta}(F) - z)^{-1} - \chi(H_0 - z)^{-1}\| = 0$,

(iii) $\lim_{F\to 0} \|K_\theta(F,z) - K(z)\| = 0$,

uniformly in $z \in \kappa$.

Proof. By using the arguments of the appendix the operator $H_0 = T_s + T_u$ on $L^2(\Omega)$ has a core given by (7.57) i.e. for $z \in \rho(H_0)$, $\mathcal{C}' = (H_0 - z)\mathcal{C}$ is dense in $L^2(\Omega)$. Let $0 < F \le F_0$ and $z \in \kappa$. For all $\varphi \in \mathcal{C}$, set $\psi = (H_0 - z)\varphi$. The resolvent equation implies,

$$(\widetilde{H}_{0,\theta}(F) - z)^{-1}\psi - (H_0 - z)^{-1}\psi = (\widetilde{H}_{0,\theta}(F) - z)^{-1}(T_s - T_{s,\theta} - \widetilde{W}_\theta(F))\varphi. \quad (5.40)$$

Clearly $\lim_{F\to 0}\|\widetilde{W}_\theta(F)\varphi\| = 0$. On the other hand we have

$$\|(T_s - T_{s,\theta})\varphi\| \le \|\partial_s G_\theta \partial_s \varphi\| + \|R_\theta \varphi\|$$

Where G_θ and R_θ are defined as in the Section 3. Evidently $\lim_{F\to 0}\|R_\theta\varphi\| = 0$. Since $\mathrm{supp}(G_\theta) = [\frac{E}{F\cos(\eta)}, \frac{E_+}{F\cos(\eta)}]$ then for such a φ, $\lim_{F\to 0}\|\partial_s G_\theta \partial_s\varphi\| = 0$. So that $\lim_{F\to 0}\|(T_s - T_{s,\theta})\varphi\| = 0$.
In view of the Lemma 4.1, $(\widetilde{H}_{0,\theta}(F) - z)^{-1}$ has a norm which is uniformly bounded w.r.t. F. Thus (i) is proved on $\mathcal{C}'$, by standard arguments then the strong convergence follows.
Let us show (ii). For $z \in \kappa$ then

$$\chi(\widetilde{H}_{0,\theta}(F) - z)^{-1} - \chi(H_0 - z)^{-1} = \chi(\widetilde{H}_{0,\theta}(F) - z)^{-1}Q_\theta(F) \tag{5.41}$$

where $Q_\theta(F) = (T_s - T_{s,\theta} - \widetilde{W}_\theta(F))(H_0 - z)^{-1}$. On $\mathrm{supp}(\chi)$, $f = 0$ then the following resolvent identity holds,

$$\chi(\widetilde{H}_{0,\theta}(F) - z)^{-1} = (H_0 - z)^{-1}\chi + (H_0 - z)^{-1}([T_s, \chi] - \chi\widetilde{W}(F))(\widetilde{H}_{0,\theta}(F) - z)^{-1}. \tag{5.42}$$

In view of (5.41) and (5.42) we have to consider two terms. First

$$t_1(F) = (H_0 - z)^{-1}\chi Q_\theta(F) = (H_0 - z)^{-1}\chi\widetilde{W}(F))(H_0 - z)^{-1}$$

which clearly converges in the norm sense to $0_{\mathcal{B}(L^2(\Omega))}$ as $F \to 0$ uniformly in $z \in \kappa$ and

$$t_2(F) = (H_0 - z)^{-1}([T_s, \chi] - \chi\widetilde{W}(F))(\widetilde{H}_{0,\theta}(F) - z)^{-1}Q_\theta(F).$$

Let $\bar{\chi}$ be the characteristic function of $\mathrm{supp}(\chi)$. We know that the operator $(H_0 - z)^{-1}\bar{\chi}$ is compact (see e.g. the proof of the Lemma 4.2) then to prove that $t_2(F)$ converges in the norm sense to $0_{\mathcal{B}(L^2(\Omega))}$ as $F \to 0$ uniformly in $z \in \kappa$, it is sufficient to show that $([T_s, \chi] - \chi\widetilde{W}(F))(\widetilde{H}_{0,\theta}(F) - z)^{-1}Q_\theta(F)$ converges strongly to $0_{\mathcal{B}(L^2(\Omega))}$ as $F \to 0$ uniformly in $z \in \kappa$. But considering the proof of (i) it is then sufficient to prove that the operator $([T_s, \chi] - \chi\widetilde{W}(F))(\widetilde{H}_{0,\theta}(F) - z)^{-1}$ is bounded operator and has a norm which is uniformly bounded w.r.t. F if F is small and $z \in \kappa$.
Evidently by the Lemma 4.1 this is true for the operator $\chi\widetilde{W}(F)(\widetilde{H}_{0,\theta}(F) - z)^{-1}$.
We have on $L^2(\Omega)$,

$$[T_s, \chi](\widetilde{H}_{0,\theta}(F) - z)^{-1} = -(\chi' g\partial_s + \partial_s g\chi')(\widetilde{H}_{0,\theta}(F) - z)^{-1} = -(2\chi' g\partial_s + (g\chi')')(\widetilde{H}_{0,\theta}(F) - z)^{-1}.$$

Since the functions g and $(g\chi')'$ are bounded and do not dependent on F, we only have to consider the operator $\chi' g^{1/2}\partial_s(\widetilde{H}_{0,\theta}(F) - z)^{-1}$.
Let $\varphi \in L^2(\Omega)$, $\|\varphi\| = 1$ set $\psi = (\widetilde{H}_{0,\theta}(F) - z)^{-1}\varphi$. Integrating by part, we have

$$\|\chi' g^{1/2}\partial_s\psi\|^2 = (-\partial_s(\chi')^2 g\partial_s\psi, \psi) \le (-\partial_s(\chi')^2 g\partial_s\psi, \psi) + (\chi' T_u\chi'\psi, \psi).$$

By using standard commutation relations, $\partial_s(\chi')^2 g\partial_s = \frac{1}{2}((\chi')^2\partial_s g\partial_s + \partial_s g\partial_s(\chi')^2 + \partial_s(g\partial_s(\chi')^2))$. Since the field $f = 0$ on $\mathrm{supp}(\chi')$ we get,

$$\begin{aligned}\|\chi' g^{1/2}\partial_s\psi\|^2 &\leq \mathrm{Re}((\widetilde{H}_{0,\theta}(F)-z)\psi,(\chi')^2\psi) - \mathrm{Re}((\widetilde{W}_\theta(F)-z)\psi,(\chi')^2\psi) + \frac{1}{2}(\partial_s(g\partial_s(\chi')^2)\psi,\psi)\\ &\leq \|(\chi')^2\|_\infty\|(\widetilde{H}_{0,\theta}(F)-z)^{-1}\| + \big(\|\partial_s(g\partial_s(\chi')^2)\|_\infty + \\ &\quad \|(\chi')^2(\widetilde{W}_\theta(F)-z)\|_\infty\big\|(\widetilde{H}_{0,\theta}(F)-z)^{-1}\|^2.\end{aligned} \tag{5.43}$$

The Lemma 4.1 implies that the l.h.s. of the last inequality is bounded uniformly w.r.t. F if F is small and $z \in \kappa$.

Note that once the strong convergence on $\mathcal{C}'$ is proved, the strong convergence on $L^2(\Omega)$ follows by using the fact that

$$\begin{gathered}([T_s,\chi] - \chi\widetilde{W}(F))(\widetilde{H}_{0,\theta}(F)-z)^{-1}Q_\theta(F) = \\ ([T_s,\chi] - \chi\widetilde{W}(F))\big((\widetilde{H}_{0,\theta}(F)-z)^{-1} - (H_0-z)^{-1}\big)\end{gathered}$$

is uniformly bounded w.r.t F for F small and $z \in \kappa$. Hence the proof of (ii) is done.

We have

$$\begin{aligned}K_\theta(F,z) - K(z) &= (V_0 + W_\theta(F) - \widetilde{W}_\theta(F))(\widetilde{H}_{0,\theta}(F)-z)^{-1} - V_0(H_0-z)^{-1}\\ &= V_0((\widetilde{H}_{0,\theta}(F)-z)^{-1} - (H_0-z)^{-1}) - (W_\theta(F) - \widetilde{W}_\theta(F))(\widetilde{H}_{0,\theta}(F)-z)^{-1}.\end{aligned}$$

Clearly in the norm sense $(W_\theta(F) - \widetilde{W}_\theta(F))(\widetilde{H}_{0,\theta}(F)-z)^{-1} \to 0_{\mathcal{B}(L^2(\Omega))}$ as $F \to 0$, uniformly w.r.t. $z \in \kappa$. By applying (ii) this is also true for $V_0((\widetilde{H}_{0,\theta}(F)-z)^{-1} - (H_0-z)^{-1})$ as $F \to 0$. Then

$$\lim_{F\to 0}\|K_\theta(F,z) - K(z)\| = 0.$$

uniformly w.r.t. $z \in \kappa$. □

5.1 Proof of the Proposition 5.3

Let E_0 be an eigenvalue of the operator H. Recall that $\lambda_0 = \inf\sigma_{ess}(H)$. Choose the reference energy, E so that $E_- = E_0 - \lambda_0 = E - \delta E$ and $\delta E = \frac{|E|}{2}$.

Let $0 < |\theta| < \theta_0, \mathrm{Im}\theta = \beta > 0$. Suppose $R > 0$ is such that the complex disk, $\mathcal{D} = \{z \in \mathbb{C}, |z - E_0| \leq R\} \subset \tilde{\nu}_\theta$ and $\mathcal{D} \cap \sigma(H) = \{E_0\}$.
First, we show that for F small enough, $z \in \partial\mathcal{D}$, $(H_\theta(F)-z)^{-1}$ exists. Clearly H has no spectrum in $\partial\mathcal{D}$ then in view of the identity

$$(H-z)^{-1} = (H_0-z)^{-1}(\mathbb{I}_{L^2(\Omega)} + K(z))^{-1},\ z \in \rho(H)\cap\rho(H_0),$$

the operator $(\mathbb{I}_{L^2(\Omega)} + K(z))^{-1}$ is well defined on $\partial\mathcal{D}$ and its norm is uniformly bounded w.r.t. $z \in \partial\mathcal{D}$.

We have

$$\mathbb{I}_{L^2(\Omega)} + K_\theta(F,z) = \Big(\mathbb{I}_{L^2(\Omega)} + (K_\theta(F,z) - K(z))(\mathbb{I}_{L^2(\Omega)} + K(z))^{-1}\Big)(\mathbb{I}_{L^2(\Omega)} + K(z)). \tag{5.44}$$

Since by the Lemma 5.3 (*iii*), $\|K_\theta(F,z) - K(z)\| \to 0$ as $F \to 0$ uniformly for $z \in \partial\mathcal{D}$, then for F small enough and $z \in \partial\mathcal{D}$

$$\|(\mathbb{I}_{L^2(\Omega)} + K(z))^{-1}(K_\theta(F,z) - K(z))\| < 1.$$

and $\mathbb{I}_{L^2(\Omega)} + (K_\theta(F,z) - K(z))(\mathbb{I}_{L^2(\Omega)} + K(z))^{-1}$ is invertible. Hence for F small enough $\mathbb{I}_{L^2(\Omega)} + K_\theta(F,z)$ is invertible for $z \in \partial\mathcal{D}$ and from (4.37), $(H_\theta(F) - z)^{-1}$ is well defined on the contour $\partial\mathcal{D}$. We define the spectral projector associated with $H_\theta(F)$,

$$P_\theta(F) = \frac{1}{2i\pi}\oint_{\partial\mathcal{D}} (H_\theta(F) - z)^{-1}\, dz. \tag{5.45}$$

The algebraic multiplicity of the eigenvalues of $H_\theta(F)$ inside $\mathcal{D}$ is just the dimension of $P_\theta(F)$. In the same way let

$$P = \frac{1}{2i\pi}\oint_{\partial\mathcal{D}} (H - z)^{-1}\, dz$$

be the spectral projector associated with H. Thus to prove the first part of the proposition, it is sufficient to show that for F small enough, $\|P_\theta(F) - P\| < 1$. We have

$$\begin{aligned}(H_\theta(F) - z)^{-1} &= (\widetilde{H}_{0,\theta}(F) - z)^{-1}(\mathbb{I}_{L^2(\Omega)} + K_\theta(F,z))^{-1} \qquad (5.46)\\ &= (\widetilde{H}_{0,\theta}(F) - z)^{-1} - (\widetilde{H}_{0,\theta}(F) - z)^{-1}K_\theta(F,z)(\mathbb{I}_{L^2(\Omega)} + K_\theta(F,z))^{-1}\end{aligned}$$

and similarly

$$(H - z)^{-1} = (H_0 - z)^{-1} - (H_0 - z)^{-1}K(z)(\mathbb{I}_{L^2(\Omega)} + K(z))^{-1}.$$

By the Lemma 4.1 the operator $\widetilde{H}_{0,\theta}(F)$ has no spectrum inside $\mathcal{D}$ this is also true for H_0 then $\oint_{\partial\mathcal{D}}(H_0 - z)^{-1}\, dz = \oint_{\partial\mathcal{D}}(\widetilde{H}_{0,\theta}(F) - z)^{-1}\, dz = 0$. Hence, we get

$$\begin{aligned}P_\theta(F) - P = \; & \tfrac{1}{2i\pi}\oint_{\partial\mathcal{D}}((H_0 - z)^{-1}K(z)(\mathbb{I}_{L^2(\Omega)} + K(z))^{-1} - \qquad (5.47)\\ & \widetilde{H}_{0,\theta}(F) - z)^{-1}K_\theta(F,z)(\mathbb{I} + K_\theta(F,z))^{-1}\, dz.\end{aligned}$$

Set $\Delta K = K(z) - K_\theta(F,z)$, $\Delta R = (H_0 - z)^{-1} - (\widetilde{H}_{0,\theta}(F) - z)^{-1}$, we have the following identity,

$$\begin{aligned}&(H_0 - z)^{-1}K(z)(\mathbb{I}_{L^2(\Omega)} + K(z))^{-1} - (\widetilde{H}_{0,\theta}(F) - z)^{-1}K_\theta(F,z)(\mathbb{I}_{L^2(\Omega)} + K_\theta(F,z))^{-1} = \\ &\Delta R K(z)(\mathbb{I}_{L^2(\Omega)} + K(z))^{-1} + (\widetilde{H}_{0,\theta}(F) - z)^{-1}(\mathbb{I}_{L^2(\Omega)} + K_\theta(F,z))^{-1}\Delta K(\mathbb{I}_{L^2(\Omega)} + K(z))^{-1}.\end{aligned}$$

By applying the Lemma 5.3 then in the norm operator sense $\Delta RK(z) \rightarrow 0_{\mathcal{B}(L^2(\Omega))}$ and $\Delta K \rightarrow 0_{\mathcal{B}(L^2(\Omega))}$ as $F \to 0$ uniformly in $z \in \partial\mathcal{D}$. Moreover the operators $(\mathbb{I}_{L^2(\Omega)} + K(z))^{-1}, (\mathbb{I}_{L^2(\Omega)} + K_\theta(F,z))^{-1}$ and $(\widetilde{H}_{0,\theta}(F) - z)^{-1}$ are uniformly bounded w.r.t. $z \in \partial\mathcal{D}$ and F for F small. This implies

$$\lim_{F \to 0} \|P_\theta(F) - P\| = 0. \tag{5.48}$$

The second part of the proposition follows from the fact that the radius of $\mathcal{D}$ can be chosen arbitrarily small, this shows that the eigenvalues of $H_\theta(F)$ inside $\mathcal{D}$ converge to E_0 as $F \to 0$.

□

6 Exponential estimates

In this section we show that the width of resonances given in the Proposition 5.3 decays exponentially when the intensity of the field $F \to 0$. Hence we prove the Theorem 2.3.

Let E_0 be an simple eigenvalue of H. For $0 < F \leq F_0'$, let Z_0 be an eigenvalue of the operator $H_\theta(F)$ in a small complex neighborhood of E_0 given by the Proposition 5.3. Then

Proposition 6.4. *Under conditions of the Theorem 5.3, there exists* $0 < F_0'' \leq F_0'$ *and two constants* $0 < c_1, c_2$ *such that for* $0 < F \leq F_0''$,

$$|\mathrm{Im} Z_0| \leq c_1 e^{-\frac{c_2}{F}}$$

First we need to prove the following lemma.

Lemma 6.4. *Let* φ_0 *be the eigenvector of* H *associated with the eigenvalue* E_0 *i.e.* $H\varphi_0 = E_0\varphi_0$. *Then there exist* $a > 0$ *such that* $e^{a|s|}\varphi_0 \in L^2(\Omega)$.

Proof. Here we use the standard Combes-Thomas argument (see e.g. [24]). Consider the following unitary transformation on $L^2(\Omega)$. Let $a \in \mathbb{R}$, for all $\varphi \in L^2(\Omega)$, set

$$W_a(\varphi)(s,u) = e^{-ias}\varphi(s,u).$$

We have

$$H_a = W_a H W_a^{-1} = H - ia(\partial_s g + g\partial_s) + ga^2.$$

The family of operators $\{H_a, a \in \mathbb{C}\}$ is an entire family of type A. Indeed it is easy to check that $D(H_a) = D(H), \forall a \in \mathbb{C}$. This follows from the fact that $\forall z \in \mathbb{C}, \mathrm{Im} z \neq 0$,

$$\|g^{1/2}\partial_s(H-z)^{-1}\| \leq \|(H-z)^{-1}\| + (\|V_0\|_\infty + |z|)\|(H-z)^{-1}\|^2.$$

Thus, for a suitable choice of z, the r.h.s of this last inequality is arbitrarily small. This implies $\partial_s g + g\partial_s$ is H-bounded with zero relative bound.

Further let $\mathrm{Re}a = 0$. Denote by $H_{0,a} = H_0 - ia(\partial_s g + g\partial_s) + ga^2$. For $\varphi \in D(H)$, we have

$$\mathrm{Re}(H_{0,a}\varphi, \varphi) = H_0 - g(\mathrm{Im}a)^2 \geq \lambda_0 - g_\infty(\mathrm{Im}a)^2;\ g_\infty = \|g\|_\infty. \tag{6.49}$$

Then for $z \notin \Sigma_a = \{z \in \mathbb{C}, \mathrm{Re}z \geq \lambda_0 - g_\infty(\mathrm{Im}a)^2\}$, $\|(H_{0,a} - z)^{-1}\| \leq \mathrm{dist}^{-1}(z, \Sigma_a)$ [20]. Thus if we show that $V_0(H_{0,a} - z)^{-1}$ is compact, then by using usual arguments of the perturbation theory (see e.g; the proof of the Proposition 4.2) the operator H_a has only discrete spectrum in $\mathbb{C} \setminus \Sigma_a$ this will imply that the essential spectrum of H_a, $\sigma_{ess}(H_a) \subset \Sigma_a$.

Let $h_0 = -\partial_s^2 \otimes \mathbb{I}_{L^2(0,d)} + \mathbb{I}_{L^2(\mathbb{R})} \otimes T_u$ be the operator introduced in the proof of the Lemma 4.2 and $G = g - 1$ we have

$$V_0(H_{0,a} - z)^{-1} = V_0(h_0 - z)^{-1} - V_0(h_0 - z)^{-1}\big(\partial_s G\partial_s + ia(\partial_s g + g\partial_s) - ga^2\big)(H_{0,a} - z)^{-1}.$$

We know that $V_0(h_0 - z)^{-1}$ is compact (see the proof of the Lemma 4.2), so we are left to show that $\big(\partial_s G\partial_s + ia(\partial_s g + g\partial_s) - ga^2\big)(H_{0,a} - z)^{-1}$ is a bounded operator. We have

$$\partial_s G\partial_s(H_{0,a} - z)^{-1} = \partial_s G\partial_s(H_0 - z)^{-1} + \partial_s G\partial_s(H_0 - z)^{-1}(ia(\partial_s g + g\partial_s) - ga^2)(H_{0,a} - z)^{-1}.$$

since $D(H_0) \subset \mathcal{H}^2_{\mathrm{loc}}(\bar{\Omega}) \cap \mathcal{H}^1_0(\Omega)$, by the closed graph theorem $\partial_s G\partial_s(H_0 - z)^{-1}$ is bounded. By using similar arguments as in the proof of the Lemma 5.3 (*ii*), $(ia(\partial_s g + g\partial_s) - ga^2)(H_{0,a} - z)^{-1}$ is also a bounded operator.

We now conclude the proof of the lemma by using usual arguments [24]. If $g_\infty(\mathrm{Im}a)^2 < \lambda_0 - E_0$, E_0 remains an discrete eigenvalue of H_a and $e^{\mathrm{Im}as}\varphi \in L^2(\Omega)$. $\square$

6.1 Proof of the Proposition 6.4

Let E_0 be a simple eigenvalue of H, as above we denote by φ_0 the associated eigenvector and $P = (., \varphi_0)\varphi_0$.

Let $\chi_1 = \chi_1(s)$ be a C^∞ characteristic function of the interval $[\frac{-\tau}{F}, \frac{\tau}{F}]$, $\tau > 0$, s.t. $\chi_1(s) = 1$ if $s \in [\frac{-\tau}{2F}, \frac{\tau}{2F}]$. Introduce the following operator on $L^2(\Omega)$,

$$H_1(F) := H + \chi_1 W(F).$$

Since $n_1 = \|\chi_1 W(F)\|_\infty < \infty$ then $H_1(F)$ is a selfadjoint operator on $D(H)$. Note that $n_1 = O(\tau) + O(F)$.

By using standard perturbation theory, we can choose $R > 0$ such that the complex disk $\mathcal{D} = \{z \in \mathbb{C}, |z - E_0| \le R\}$ such that $\mathcal{D} \cap \sigma(H) = \{E_0\}$ and has a boundary $\partial\mathcal{D} \subset \rho(H_1(F)$ for τ and F small enough. Then

$$P_1 = P_1(F) = \frac{1}{2i\pi} \oint_{\partial\mathcal{D}} (H_1(F) - z)^{-1}\, dz. \tag{6.50}$$

is an spectral projector for $H_1(F)$ satisfying

$$\lim_{\tau\to 0, F\to 0} \|P_1 - P\| \to 0. \tag{6.51}$$

Hence for τ and F small enough, the operator $H_1(F)$ has one eigenvalues near E_0, $e_0(F)$ and $|E_0 - e_0(F)| = O(\tau) + O(F)$. Denote by ψ_0 the associated eigenvector. Evidently $P_1\psi_0 = \psi_0$.

Let us show that as a consequence of the Lemma 6.4, if F and τ are small enough then $e^{a|s|}\psi_0 \in L^2(\Omega)$ and

$$\|e^{a|s|}\psi_0\| \le C \tag{6.52}$$

where the constant $C > 0$ and it is independent of F.
Introduce the family of operators $H_{1,a} = H_a + \chi_1 W(F)$, where H_a is defined as in the previous section. Then $\{H_{1,a}, a \in \mathbb{C}\}$ is an entire family of type A. In the other hand the spectrum of $H_{1,a}$ satisfies, $\sigma(H_{1,a}) \subset \{z \in \mathbb{C}, dist(z, \sigma(H_a) \le n_1\}$.

We have

$$(\varphi_0, \psi_0)e^{as}\psi_0 = \frac{1}{2i\pi} \oint_{\partial\mathcal{D}} e^{as}(H_1(F) - z)^{-1}e^{-as}e^{as}\varphi_0\, dz. \tag{6.53}$$

For τ and F small enough, the resolvent $(H_{1,a}(F) - z)^{-1}$ is well defined for any $z \in \partial\mathcal{D}$. Further, the resolvent identity

$$(H_{1,a}(F) - z)^{-1} = (H_a - z)^{-1} - (H_a - z)^{-1}\chi_1 W(F)(H_{1,a}(F) - z)^{-1}$$

and the fact that $\|(H_a - z)^{-1}\|$ is uniformly bounded in $z \in \partial\mathcal{D}$ imply that $\|(H_{1,a}(F) - z)^{-1}\|$ is uniformly bounded in $z \in \partial\mathcal{D}$ w.r.t. τ and F.
Moreover by using standard arguments, in the bounded operator sense $(H_{1,a}(F) - z)^{-1} = e^{as}(H_1(F) - z)^{-1}e^{-as}$ for $z \in \partial\mathcal{D}$. In the other hand we can check that $|(\varphi_0, \psi_0)| \ge \frac{1}{2}$ if F and τ are chosen small enough. Hence by using the Lemma 6.4 and (6.53), there exists $C > 0$ independent of τ and F such that

$$\|e^{as}\psi_0\| \le C\|e^{as}\varphi_0\| < \infty.$$

The same arguments can be applied with a changing in $-a$, proving our claim.
From now, we fix $\tau > 0$ and we choose $0 < F < F_0$ where F_0 is small enough such that (6.52) also holds.

Let $\theta = i\beta$, $0 < \beta < \theta_0$. As in previous section $P_1 = P_1(F), P_\theta = P_\theta(F)$ are the spectral projectors of $H_1 = H_1(F), H_\theta = H_\theta(F)$ associated respectively to the eigenvalue e_0, Z_0. We have

$$(Z_0 - e_0)(P_\theta\psi_0, P_1\psi_0) = ((H_\theta - H_1)P_\theta\psi_0, P_1\psi_0) = \\ ((\theta F\cos(\eta)f + (1-\chi_1)W(F) + \Delta T P_\theta\psi_0, P_1\psi_0)$$

where $\Delta T = T_{s,\theta} - T_s$. Hence we will use the estimate,

$$|\mathrm{Im}Z_0| \le \frac{1}{(|P_\theta\psi_0, P_1\psi_0)|}|(\theta F\cos(\eta)f + (1-\chi_1)W(F) + \Delta T P_\theta\psi_0, P_1\psi_0)| \quad (6.54)$$

By using (5.48), (6.51), for F and τ small enough, the l.h.s. of (6.54) is estimated as,

$$|(P_\theta\psi_0, P_1\psi_0)| \ge \frac{1}{2},$$

and from (6.52), the two first terms of the r.h.s. of (6.54) satisfy

$$|(\theta F\cos(\eta)fP_\theta\psi_0,\ P\psi_0)| \le |\theta|\|\Phi\psi_0\| = O(e^{-\frac{c}{F}})$$

and

$$|(1-\chi_1)W(F)P_\theta\psi_0,\ P\psi_0)| \le \|(1-\chi_1)W(F)\psi_0\| = O(e^{-\frac{c}{F}}),$$

for some constant $c > 0$. Let χ be a characteristic function of supp(f'). Then (see e.g (3.17) and (3.19)),

$$|(\Delta T P_\theta\psi_0, P\psi_0)| = |(\Delta T P_\theta\psi_0, \chi P\psi_0)| \le \|\chi\psi_0\|\|\Delta T P_\theta\psi_0\|.$$

Since for F small enough, $\|\chi\psi_0\| = O(e^{-\frac{c}{F}})$. Then to prove the theorem we need to show that funder our conditions, $\|\Delta T P_\theta\varphi_0\|$ and then by (5.45) that $\|\Delta T(H_\theta(F) - z)^{-1}\|$, $z \in \partial\mathcal{D}$ is uniformly bounded w.r.t. F.

Note that following the proof of the Proposition 5.3, (see e.g. (5.44) and (5.46)) then for F small enough, the norm $\|(H_\theta(F) - z)^{-1}\|, z \in \partial\mathcal{D}$ is uniformly bounded in F. Evidently this is also true for $\|(H - z)^{-1}\|$. The second resolvent equation implies for F small and $z \in \partial\mathcal{D}$,

$$\Delta T(H_\theta(F) - z)^{-1} = \\ \Delta T(H - z)^{-1} - \Delta T(H - z)^{-1}\big(\Delta T + W_\theta(F)\big)(H_\theta(F) - z)^{-1}. \quad (6.55)$$

By the closed graph theorem the operator $\Delta T(H - z)^{-1}, z \in \partial\mathcal{D}$ is bounded and if F is assumed small enough $\|\ \Delta T(H - z)^{-1}\| < \frac{1}{2}$ uniformly in $z \in \partial\mathcal{D}$ (see e.g.

the proof of the Theorem 3.1). In view of

$$\begin{gathered}\Delta T(H-z)^{-1}W_\theta(F)(H_\theta(F)-z)^{-1} = \\ \Delta T(Fs+i)(H-z)^{-1}\tfrac{W_\theta(F)}{Fs+i}(H_\theta(F)-z)^{-1} + \\ F\Delta T(H-z)^{-1}(g\partial_s+\partial_s g)(H-z)^{-1}\tfrac{W_\theta(F)}{Fs+i}(H_\theta(F)-z)^{-1},\end{gathered} \tag{6.56}$$

the same arguments already used in the Section 3, then imply that there exists a constant $C>0$ such that for F small enough $\|\Delta T(H-z)^{-1}W_\theta(F)(H_\theta(F)-z)^{-1}\| \leq C$ for $z \in \partial\mathcal{D}$. Therefore, by (6.55), we get for $z \in \partial\mathcal{D}$,

$$\begin{gathered}\|\Delta T(H_\theta(F)-z)^{-1}\|(1-\|\Delta T(H-z)^{-1}\|) \leq \|\Delta T(H-z)^{-1}\| + \\ \|\Delta T(H-z)^{-1}W_\theta(F)(H_\theta(F)-z)^{-1}\|,\end{gathered}$$

hence we get

$$\|\Delta T(H_\theta(F)-z)^{-1}\| \leq 1+2C.$$

□

7 Appendix: Self-adjointness

In this section we prove the Theorem 2.1. Our proof is mainly based on the commutator theory [23, 25]. First we note that it is sufficient to show the theorem for the operator $h = h(F) = H_0 + w(F)$ defined on $L^2(\Omega)$ where $w(F)$ is defined in (3.18). Choose $a, b \in \mathbb{R}^+$ such that $w(F,s) + as^2 + b > 1$ and consider the positive symmetric operator in $L^2(\Omega)$,

$$N = H_0 + w(F) + 2as^2 + b.$$

The operator N admits a (Friedrichs) self-adjoint extension since it is associated with a positive quadratic form, we denote its self-adjoint extension by the same symbol [20]. Moreover N has compact resolvent and then only discrete spectrum (see section 7.1 below). So N is essentially self-adjoint on

$$\mathcal{C} = \{\varphi = \psi_{|\mathbf{\Omega}} : \psi \in \mathrm{S}(\mathbb{R}^2), \psi(s,0) = \psi(s,d) = 0 \text{ for all } s \in \mathbb{R}\} \tag{7.57}$$

where $\mathcal{S}(\mathbb{R}^2)$ denotes the Schwartz class. In fact $\mathcal{C}$ contains a complete set of eigenvectors of N. Indeed some standard arguments (see e.g. [3, 13, 24]) show that the corresponding eigenfunctions and their derivatives are smooth on $\bar{\Omega}$ and super-exponentially decay in the longitudinal direction. From [23, X.5] we have to check that there exist $c, d > 0$ such that for all $\varphi \in \mathcal{C}$, $\|\varphi\| = 1$,

$$c\|N\varphi\| \geq \|h\varphi\| \tag{7.58}$$

and

$$d\|N^{\frac{1}{2}}\varphi\|^2 \geq |(h\varphi, N\varphi) - (N\varphi, h\varphi)|. \tag{7.59}$$

In the quadratic forms sense on $\mathcal{C}$,

$$N^2 \;=\; (h+b)^2 + 4asNs + [[h,s],s]. \tag{7.60}$$

But in the form sense on $\mathcal{C}$, $[[h,s],s] = -2g$ and g is bounded function. Therefore,

$$\|N\varphi\| + 2\|g\|_\infty \geq \|(h+b)\varphi\|$$

and then since $N \geq 1$ this last inequality implies (7.58). Similarly,

$$\begin{aligned} \pm i[h,N] &= \pm i[h-N,N] = \pm i2a[s^2, T_s] \\ &= \mp i4a(\partial_s g s + s g \partial_s), \end{aligned}$$

this gives that for all $\varphi \in \mathcal{C}$, $\|\varphi\| = 1$,

$$|(h\varphi, N\varphi) - (N\varphi, h\varphi)| \leq 2a(\|g^{\frac{1}{2}}\partial_s\varphi\|^2 + \|sg^{\frac{1}{2}}\varphi\|^2). \tag{7.61}$$

Clearly we have $N \geq T_s + as^2$ on $\mathcal{C}$. Then from (7.61) there exists a constant $d > 0$ such that

$$|(h\varphi, N\varphi) - (N\varphi, h\varphi)| \leq d(N\varphi, \varphi)$$

proving (7.59).

We now show *(ii)*. Let $E \in \mathbb{R}$. We denote by $\tilde{E}_1$ the first eigenvalue of the operator $T_u + F\sin(\eta)u$ and $\tilde{\chi}_1$ the associated normalized eigenvector,

$$(T_u + F\sin(\eta)u)\tilde{\chi}_1(u) = \tilde{E}_1\tilde{\chi}_1(u). \tag{7.62}$$

Set $\lambda = E - \tilde{E}_1$ and φ be the solution of the Airy equation

$$-\varphi''(s) + F\cos(\eta)s\varphi(s) = \lambda\varphi(s) \;\; \lambda \in \mathbb{R}. \tag{7.63}$$

It is known (see e.g. [1]) that $\varphi(s) = (\lambda - F\cos(\eta)s)^{-1/4}e^{-i\frac{2}{3F\cos(\eta)}(\lambda - F\cos(\eta)s)^{3/2}} + o((\lambda - F\cos(\eta)s)^{-1/4})$ and $\varphi'(s) = (\lambda - F\cos(\eta)s)^{1/4}e^{-i\frac{2}{3F\cos(\eta)}(\lambda - F\cos(\eta)s)^{3/2}} + o((\lambda - F\cos(\eta)s)^{1/4})$ as $s \to -\infty$.
Let ξ be a C^∞ characteristic function of $(-1,1)$ and $s \in \mathbb{R} \to \xi_n(s) = \xi(\frac{s}{n^\alpha} + n)$, $\frac{1}{2} < \alpha < 1, n \in \mathbb{N}^*$. Set

$\psi_n = \frac{\widetilde{\psi}_n}{\|\widetilde{\psi}_n\|}$ where $\widetilde{\psi}_n(s,u) = \tilde{\chi}_1(u)\varphi(s)\xi_n(s)$, then for n large enough, $\|\widetilde{\psi}_n\| = \|\varphi\xi_n\| \geq c\, n^{\alpha/2-1/4}$ for some constant $c > 0$. Since $g = 1$ if n is large, we have

$$(H(F) - E)\psi_n = \big(-2\chi_1(u)\varphi'(s)\xi_n'(s) - \chi_1(u)\varphi(s)\xi_n''(s)\big)\frac{1}{\|\widetilde{\psi}_n\|}$$

and then

$$\|(H(F)-E)\psi_n\|_{L^2(\Omega)} \leq \frac{1}{\|\widetilde{\psi}_n\|}\big(2\|\varphi'\xi_n'\|_{L^2(\mathbb{R})} + \|\varphi\xi_n''\|_{L^2(\mathbb{R})}\big). \tag{7.64}$$

For n large enough $\|\varphi'\xi_n'\|^2_{L^2(\mathbb{R})} = o(n^{-\alpha/2+1/4})$ and $\|\varphi\xi_n''\|_{L^2(\mathbb{R})} = o(n^{-3\alpha/2-1/4})$. Thus,

$$\lim_{n\to\infty} \|(H(F)-E)\psi_n\|_{L^2(\Omega)} = 0.$$

This completes the proof.

□

7.1 The operator $(N+1)^{-1}$

Consider first the positive self-adjoint operator on $L^2(\Omega)$

$$N_0 = (-\partial_s^2 + v(s)) \otimes \mathbb{I}_{L(0,d)} + \mathbb{I}_{L(\mathbb{R})} \otimes T_u$$

where $v(s) = w(F,s) + 2as^2 + b$ and w is defined in (3.18). It is known that the operator $-\partial_s^2 + v(s)$ is essentially self-adjoint on $L^2(R)$ and has a compact resolvent [23, 24]. By the min-max principle we can verify that the eigenvalues of this operator satisfy, there exists $c_1, c_2 > 0$ such that for large $n \in \mathbb{N}$

$$c_1 n \leq e_n \leq c_2 n.$$

Then $(N_0+1)^{-1}$ is an Hilbert-Schmidt operator. By using the second resolvent equation we have

$$(N+1)^{-1} = (N_0+1)^{-1} + (N_0+1)^{-1}\partial_s G \partial_s (N+1)^{-1}$$

where G is defined in the proof of the Lemma 4.2. Therefore, the statement follows if we show that $\partial_s G \partial_s (N+1)^{-1}$ is a bounded operator.

We have

$$\partial_s G \partial_s (N+1)^{-1} = \partial_s G \partial_s (H_0+1)^{-1} - \partial_s G \partial_s (H_0+1)^{-1} v (N+1)^{-1}.$$

Since $D(H_0) \subset \mathcal{H}^2_{\text{loc}}(\bar{\Omega}) \cap \mathcal{H}^1_0(\Omega)$, by the closed graph theorem $\partial_s G \partial_s (H_0+1)^{-1}$ and $\partial_s G \partial_s s (H_0+1)^{-1}$ are bounded. Standard commutation relations then imply,

$$\begin{gathered}\partial_s G \partial_s (H_0+1)^{-1} v (N+1)^{-1} = \partial_s G \partial_s (s+i)(H_0+1)^{-1} \tfrac{v}{s+i}(N+1)^{-1} + \\ \partial_s G \partial_s (H_0+1)^{-1} 2\partial_s (H_0+1)^{-1} \tfrac{v}{s+i}(N+1)^{-1}.\end{gathered}$$

We know that the domain $D(N) \subset D(|v|^{1/2})$ so $\frac{v}{s+i}(N+1)^{-1}$ is bounded, then it follows by using the same arguments as above that $\partial_s G \partial_s (H_0+1)^{-1} v (N+1)^{-1}$ is also bounded. □

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